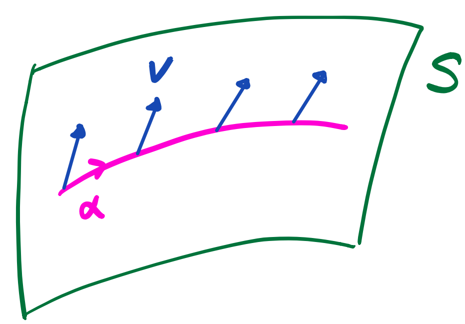
§ Parallel Transport

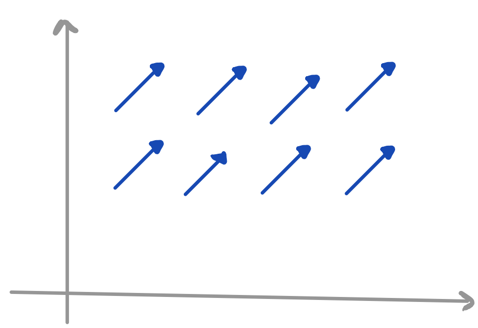
Let $\alpha : [a,b] \longrightarrow S$ be a curve on S.

Suppose V is a tangential vector field defined on α.

Question: When do we consider V as "parallel" along α ?



Note: The concept of "parallel" is clear in $\mathbb{R}^n$ since we can translate vectors from a point to any other point.



a parallel vector field in $\mathbb{R}^2$

BUT there is no "translations" on a surface S !

So we need to make a definition:

Def^{n}: A tangential vector field V is parallel along a curve $\alpha : [a,b] \to S$ on a surface S if

$$\boxed{\nabla_{\alpha'} V \equiv 0}$$

i.e. V is constant as seen intrinsically on the surface.

Prop: If V_1, V_2 are two parallel tangential vector fields along a curve α on S, then

$$\boxed{\langle V_1, V_2 \rangle \equiv \text{constant}}$$

Proof: Let $\alpha(t) : [a,b] \to S$. Then we can think of $\langle V_1, V_2 \rangle(t) = \langle V_1(\alpha(t)), V_2(\alpha(t)) \rangle$ as a function of t. Using metric compatibility of ∇

$$\frac{d}{dt}\langle V_1, V_2 \rangle = \langle \underbrace{\nabla_{\alpha'} V_1}_{= 0}, V_2 \rangle + \langle V_1, \underbrace{\nabla_{\alpha'} V_2}_{= 0} \rangle = 0$$

$\square$

Cor: (1) A parallel vector field has constant length.

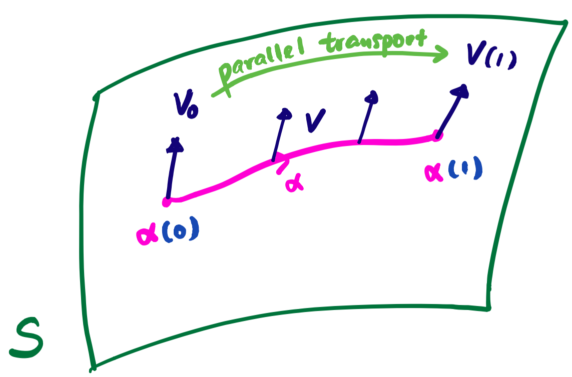
(2) Two parallel vector fields have constant angle between them.

Now, having the concept of "parallelism" along curves, we can move a vector along a given curve in such a way that the vector is "unchanged" as seen on S.

Thm: Let $\alpha : [0,1] \to S$ be a curve on S. For any given $V_0 \in T_{\alpha(0)}S$, there exists a unique parallel tangential vector field V defined along α s.t.

$$V(0) = V_0 .$$

Note: The vector $V(1) \in T_{\alpha(1)}S$ is said to be the parallel transport of V_0 from $\alpha(0)$ to $\alpha(1)$ along the curve α.



$$\nabla_{\alpha'} V \equiv 0$$

Proof: We first express the parallel condition $\nabla_{\alpha'}V \equiv 0$ in local coordinates (u^1, u^2). Let $\alpha(t) = (u^1(t), u^2(t))$.

$$\alpha'(t) = \frac{du^1}{dt}\partial_1 + \frac{du^2}{dt}\partial_2$$

$$V(t) = v^1(t)\,\partial_1 + v^2(t)\,\partial_2$$

and $$\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k\,\partial_k .$$

Therefore, we have

$$\begin{aligned}
\nabla_{\alpha'}V &= \nabla_{\frac{du^i}{dt}\partial_i}(v^j\partial_j) \\
&= \frac{du^i}{dt}(\partial_i v^j)\,\partial_j + \frac{du^i}{dt}v^j\,\nabla_{\partial_i}\partial_j \\
&= \frac{du^i}{dt}(\partial_i v^j)\,\partial_j + \frac{du^i}{dt}v^j\,\Gamma_{ij}^k\,\partial_k \\
&= \left(\frac{d}{dt}v^k + \frac{du^i}{dt}\Gamma_{ij}^k v^j\right)\partial_k
\end{aligned}$$

Hence, $\nabla_{\alpha'}V \equiv 0$ is equivalent to the following linear 1st order system of ODEs for the unknowns v^1, v^2:

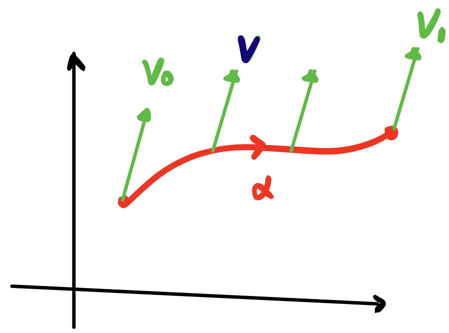
$$(*)\quad \boxed{\frac{dv^k}{dt} + \frac{du^i}{dt}\Gamma_{ij}^k v^j = 0} \qquad (k = 1, 2)$$

which is uniquely solvable given $v^1(0), v^2(0)$.

$\blacksquare$

Examples:

(1) Plane

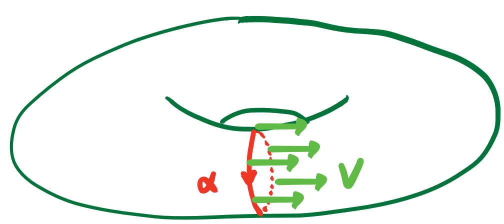


The parallel vector field is

$$V \equiv V_0$$

(2) Torus of revolution

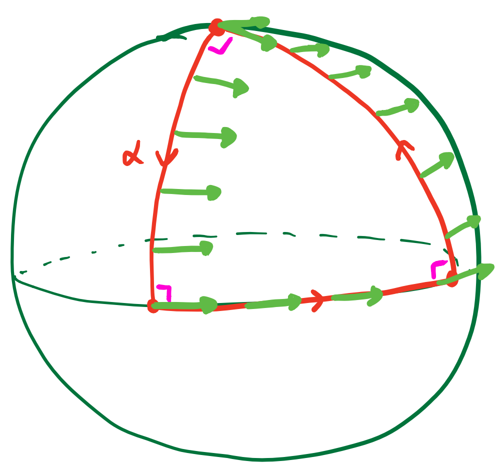
α : a meridian



Note: Parallel transport of a vector around the loop α returns to the same vector.

(Not true in general!)

(3) Sphere



Note: Parallel transport of a vector around this closed loop α gives a different vector than the vector we begin with!

§ Geodesics

Def^n: A curve $\alpha : [a,b] \to S$ is said to be a **geodesic** on the surface S if

$$\boxed{\nabla_{\alpha'} \alpha' \equiv 0}$$

Note: In other words, the tangent vector field α' is parallel along α. From our discussion above,

$$\| \alpha' \| \equiv \text{constant}$$

i.e. any geodesic α is automatically parametrized proportional to arc length.

Prop: α is geodesic if and only if in any local coordinate system $\alpha(t) = (u^1(t), u^2(t))$, we have

$$(\#) \quad \boxed{\frac{d^2 u^k}{dt^2} + \Gamma^k_{ij}(\alpha(t)) \frac{du^i}{dt} \frac{du^j}{dt} = 0} \qquad k = 1, 2$$

Proof: Plug $v^i = \frac{du^i}{dt}$ into $(*)$.

Remark: $(\#)$ is a system of 2^{nd} order, non-linear ODEs.

By standard ODE theory, we have the following:

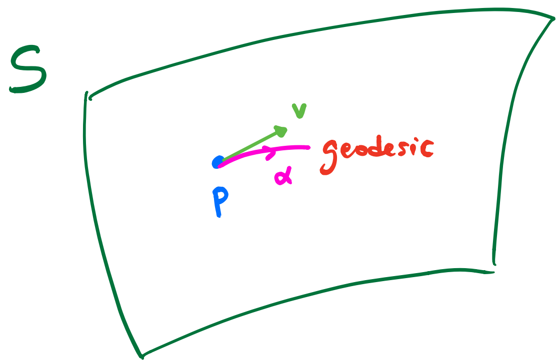
Theorem: Let $S \subseteq \mathbb{R}^3$ be a surface.

For any given $p \in S$ and $v \in T_pS$,

there exists $\varepsilon > 0$ and a unique geodesic

$$\alpha : [0, \varepsilon) \longrightarrow S$$

s.t. $\alpha(0) = p$, $\alpha'(0) = v$.



"Short time existence & uniqueness for geodesics"

As an example, we "compute" the geodesics lying on a plane in two different coordinate systems:

① Rectangular coordinates

$X_{rec}(x, y) = (x, y, 0)$

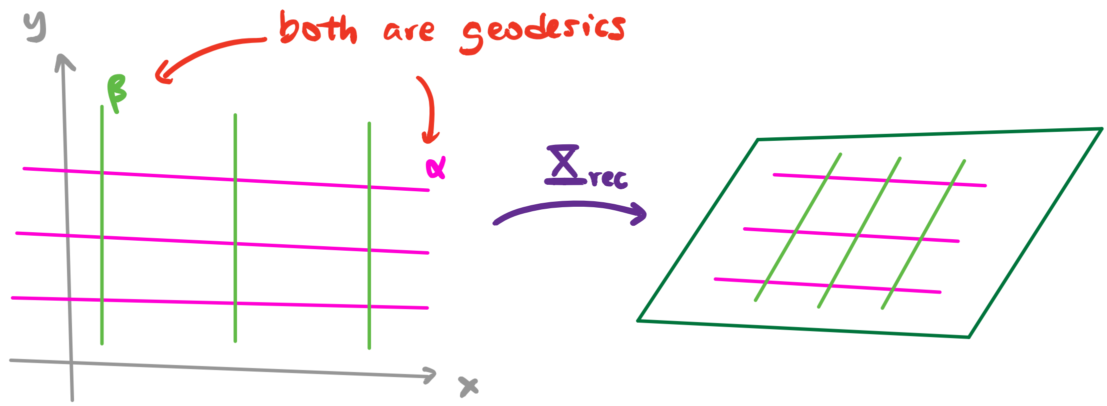
$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Gamma_{ij}^k = 0 \quad \forall\, i, j, k$

(#) becomes:

$$\begin{cases} \dfrac{d^2x}{dt^2} = 0 \\[2mm] \dfrac{d^2y}{dt^2} = 0 \end{cases} \Rightarrow$$

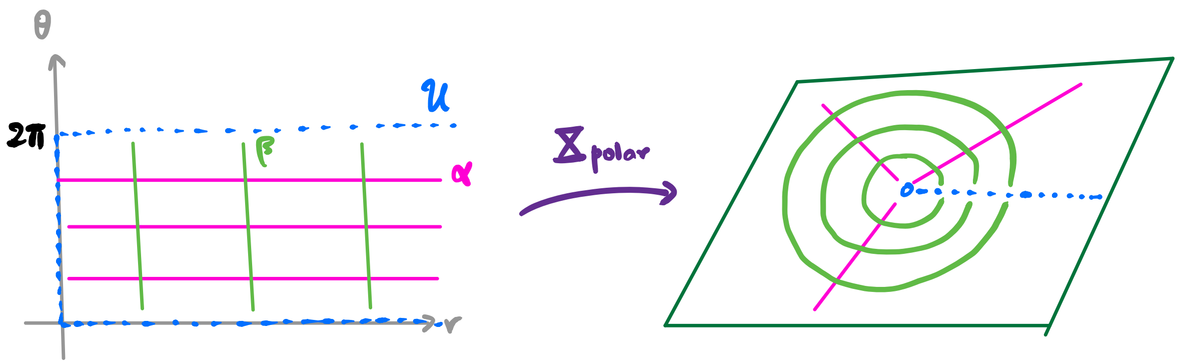
$x(t)$, $y(t)$ are linear functions of t.



② Polar coordinates

$$\mathbf{X}_{polar}(r,\theta) = (r\cos\theta,\ r\sin\theta,\ 0)$$

where $r > 0$, $0 < \theta < 2\pi$



$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

$$\begin{cases} \Gamma^r_{rr} = \Gamma^r_{r\theta} = \Gamma^\theta_{rr} = \Gamma^\theta_{\theta\theta} = 0 \\ \Gamma^r_{\theta\theta} = -r \\ \Gamma^\theta_{r\theta} = \frac{1}{r} \end{cases}$$

(#) becomes:

$$\boxed{\begin{aligned} r'' - r(\theta')^2 &= 0 \\ \theta'' + \frac{2}{r} r'\theta' &= 0 \end{aligned}} \quad (\star)$$

α : $\theta \equiv$ const. , $r(t) = At + B$ solves $(*)$

hence they are geodesics!

β : $r \equiv$ const. , $\theta(t) = At + B$ does <u>NOT</u> solve $(*)$

(unless $A = 0$, degenerate!)

hence they are <u>NOT</u> geodesics!

§ Geodesics on Surfaces in $\mathbb{R}^3$

<u>Recall</u>: A curve $\alpha : I \to S \subseteq \mathbb{R}^3$ is a geodesic

iff $\quad \nabla_{\alpha'} \alpha' \equiv 0$

i.e. $\quad (\alpha'')^T = \nabla_{\alpha'} \alpha' \equiv 0$.

In local coordinates, it can be expressed a system of non-linear 2nd order ODEs:

$$\boxed{\frac{d^2 u^k}{dt^2} + \Gamma^k_{ij} \frac{du^i}{dt} \frac{du^j}{dt} = 0} \quad \text{— } (*)$$

<u>Fundamental Theorem of geodesics:</u>

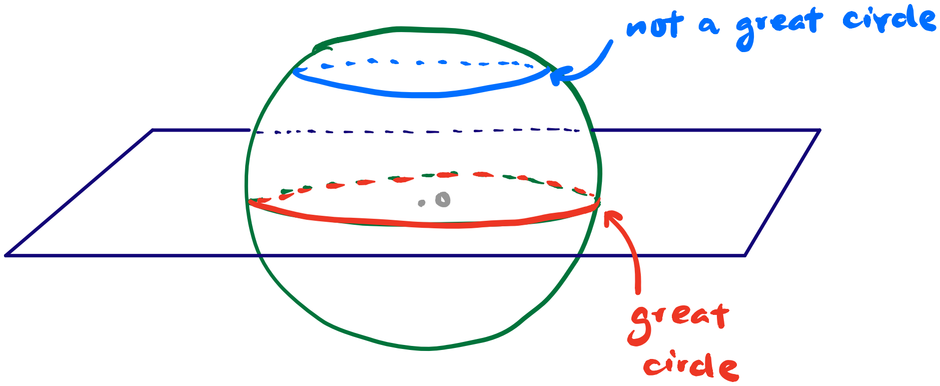
$(*)$ is uniquely solvable (on a short time interval) with any prescribed initial position and initial velocity.

<u>Remark</u>: However, explicit solutions can be very hard to compute analytically. We now look at a few simple examples that allow us to simplify the calculations by making use of the symmetries!

Example 1: Geodesics on round spheres.

Let $S = \mathbb{S}^2 = \{x^2+y^2+z^2=1\}$ be the unit sphere.

Prop: The geodesics on $\mathbb{S}^2$ are given exactly by (segments) of the "great circles", i.e. circles obtained by intersecting $\mathbb{S}^2$ with a plane passing through the origin.



Proof: Suppose $\alpha(s): I \to \mathbb{S}^2$ is a geodesic p.b.a.l.

α lies on $\mathbb{S}^2 \Rightarrow \|\alpha\|^2 \equiv 1$ ...... (1)

α p.b.a.l. $\Longrightarrow \|\alpha'\|^2 \equiv 1$ ...... (2)

α geodesic $\Longrightarrow (\alpha'')^T \equiv 0$ ...... (3)

Differentiate (1), $\langle \alpha, \alpha' \rangle \equiv 0$

Differentiate this again and use (2) :

$$\langle \alpha, \alpha'' \rangle + \underbrace{\langle \alpha', \alpha' \rangle}_{\substack{\parallel \\ 1}} \equiv 0$$

$$\Rightarrow \qquad \langle \alpha'', \alpha \rangle \equiv -1$$

<u>Recall that</u> : $T_p S^2 \perp p$

$$\therefore \quad \alpha'' = (\alpha'')^T + (\alpha'')^N$$

$$= \underbrace{(\alpha'')^T}_{\substack{\text{geodesic} \quad \parallel \\ 0}} + \underbrace{\langle \alpha'', \alpha \rangle}_{\substack{\parallel \\ -1}} \alpha$$

Hence, we arrive at the equation : $\boxed{\alpha'' + \alpha \equiv 0}$ —— (#)

<u>Note</u>: $\boxed{\alpha(s) = p \cos s + q \sin s}^{*}$ solves (#)

for any $p, q \in S^2$ s.t. $p \perp q$ (Exercise : Check this)

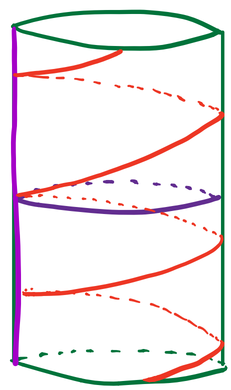
By choosing p, q appropriately, it solves (#) with any given initial position $\alpha(0)$ and initial velocity $\alpha'(0)$.

Hence, by uniqueness, these are all the possible solutions to the geodesic equation. Finally, notice that * parametrizes a great circle lying in the plane ($\subset S^2$) spanned by p and q.

$\square$

Example 2: Geodesics on a cylinder

Let $S = \{x^2+y^2=1\}$ be a right circular cylinder (of radius 1)



Prop: The geodesics are given by segments of either
- horizontal circle
- vertical line
- helix

We will give 2 proofs of this.

Proof 1: (Make use of symmetry)

As before, let $\alpha : I \to S$ be a geodesic p.b.a.l.

Suppose $\alpha(s) = (x(s), y(s), z(s))$, $s \in I$

α lies on $S \Rightarrow x(s)^2 + y(s)^2 \equiv 1$

$\xRightarrow{\text{differentiate}} xx' + yy' \equiv 0$

$\xRightarrow{\text{differentiate}} xx'' + yy'' + \underbrace{(x')^2 + (y')^2}_{= 1 - (z')^2 \ (\because \text{p.b.a.l.})} \equiv 0.$

Hence, $\boxed{xx'' + yy'' = (z')^2 - 1}$ — (##)

Recall: $T_{(x,y,z)}S \perp (x,y,0)$

geodesic equation: $(\alpha'')^T \equiv 0 \Rightarrow (x'', y'', z'') \parallel (x, y, 0)$

In other words, $\exists$ function $\lambda(s)$ s.t.

$$\begin{cases} x''(s) = \lambda(s)\, x(s) \\ y''(s) = \lambda(s)\, y(s) \\ z''(s) = 0 \end{cases} \qquad \text{———} \quad (**)$$

Now, we solve $(**)$ with initial conditions:

$$\alpha(0) = (x(0), y(0), z(0)) = (1, 0, 0)$$

$$\alpha'(0) = (x'(0), y'(0), z'(0)) = (0, a, b) \perp \alpha(0)$$

where $a^2 + b^2 = 1$ p.b.a.l.

Solving for z, we have $z(s) = bs$ $(\Rightarrow z' \equiv b)$

Put everything back into $(\#\#)$

$$\lambda(s) = x(s)x''(s) + y(s)y''(s) = b^2 - 1 = -a^2.$$

$(\because x^2 + y^2 \equiv 1)$

Constant!

Solve $x'' = -a^2 x$ with $x(0) = 1$, $x'(0) = 0$

$$x(s) = \cos as$$

Solve $y'' = -a^2 y$ with $y(0) = 0$, $y'(0) = a$

$$y(s) = \sin as$$

In summary, we have

$$\alpha(s) = (\cos as,\ \sin as,\ bs) \qquad s \in I$$

where $a, b \in \mathbb{R}$ are constants s.t. $a^2 + b^2 = 1$.

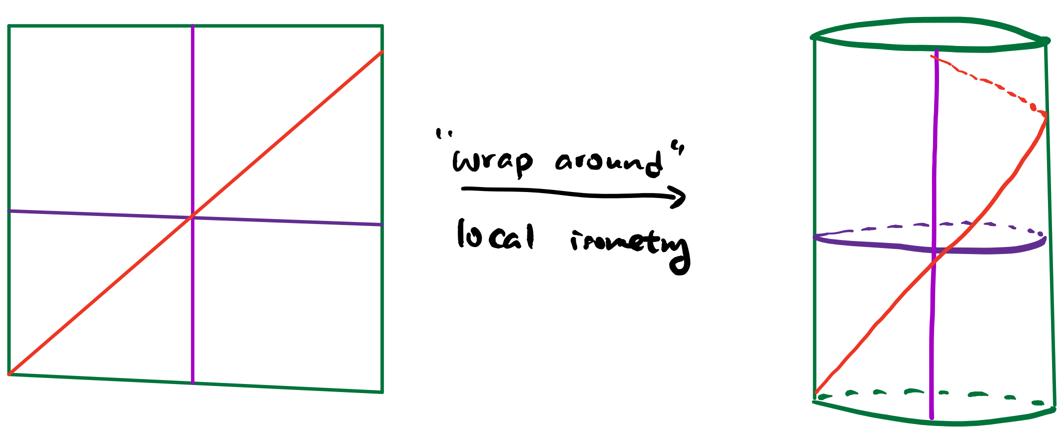
Case 1: $b = 0,\ a = 1 \Rightarrow$ horizontal circle

Case 2: $b = 1,\ a = 0 \Rightarrow$ vertical line

Case 3: $b \neq 0,\ a \neq 0 \Rightarrow$ helix

$\square$

Proof 2: Geodesics are intrinsic concepts, thus is preserved by (local) isometries.



$\square$

§ First Variation Formula for arc length

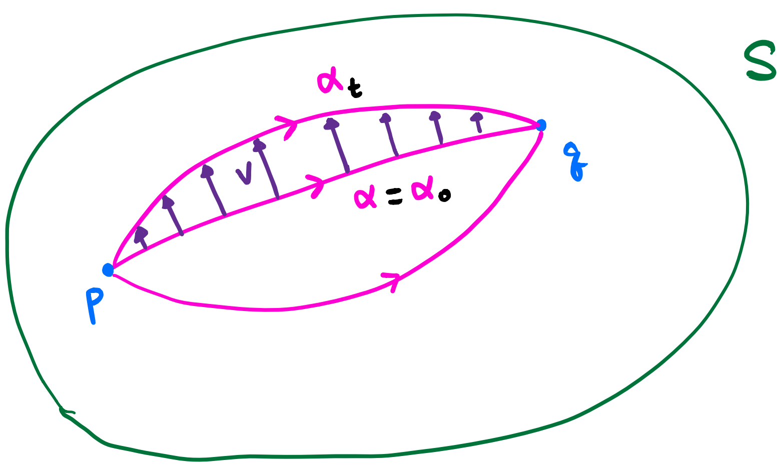
Let $\alpha : [0, L] \to S$ be a curve on S p.b.a.l.

with end points $p = \alpha(0)$, $q = \alpha(L)$.

Consider a "variation" of α with fixed endpoints:

$$\{\alpha_t(s)\}_{t \in (-\varepsilon, \varepsilon)} \quad \text{s.t.} \quad \alpha_0 = \alpha$$

$\alpha_t : [0, L] \longrightarrow S$ and $\alpha_t(0) = \alpha_0(0)$, $\alpha_t(L) = \alpha_0(L)$ for all $t \in (-\varepsilon, \varepsilon)$



where

$$V(s) = \frac{d}{dt}\Big|_{t=0} \alpha_t(s)$$

variation vector field

(tangent to S)

First Variation Formula:

$$\boxed{\frac{d}{dt}\Big|_{t=0} \mathrm{Length}(\alpha_t) = -\int_0^L \langle V(s), \alpha''(s) \rangle \, ds}$$

Remark: If α is a critical point for <u>ALL</u> variations,

then
$$0 = \frac{d}{dt}\Big|_{t=0} \text{Length}(\alpha_t) = -\int_0^L \langle V(s), \alpha''(s) \rangle \, ds$$

for <u>ALL</u> variation field V

(which is tangential)

$$\Longrightarrow \quad \boxed{(\alpha'')^T \equiv 0}$$

Hence, geodesics are exactly the critical points to the arc length functional!

Proof of 1st Variation Formula:

By definition of arc length, (<u>Note:</u> $\alpha_t(s)$ is <u>NOT</u> p.b.a.l. except for $t=0$)

$$\text{Length}(\alpha_t) = \int_0^L \| \alpha_t'(s) \| \, ds$$

Differentiate w.r.t. t.

$$\frac{d}{dt} \text{Length}(\alpha_t) = \int_0^L \frac{1}{\| \alpha_t'(s) \|} \left\langle \frac{\partial}{\partial t}\left(\frac{\partial \alpha}{\partial s}\right), \frac{\partial \alpha}{\partial s} \right\rangle ds$$

switch

$$= \int_0^L \frac{1}{\| \alpha_t'(s) \|} \left\langle \frac{\partial}{\partial s}\left(\frac{\partial \alpha}{\partial t}\right), \frac{\partial \alpha}{\partial s} \right\rangle ds$$

Evaluate at $t=0$, $\| \alpha_0'(s) \| \equiv 1$, $\frac{\partial \alpha}{\partial t}\Big|_{t=0} = V$

$$\frac{d}{dt}\Big|_{t=0} \text{Length}(\alpha_t) = \int_0^L \left\langle \frac{\partial}{\partial s} V, \frac{\partial}{\partial s} \alpha \right\rangle ds$$

$$= -\int_0^L \langle V, \alpha'' \rangle ds$$

$$+ \cancel{\left\langle V, \frac{\partial}{\partial s} \alpha \right\rangle \Big|_{s=0}^{s=L}} \quad = 0$$

(fixed end points)

$\square$